Exam

26/01/2023, 8:30 am - 10:30 am

Instructions:

- Prepare your solutions in an **ordered, clear and clean way**. Avoid delivering solutions with scratches.
- Write your name and student number in **all** pages of your solutions.
- Clearly indicate each exercise and the corresponding answer. Provide your solutions with as much detail as possible.
- Use different pieces of paper for solutions of different exercises.
- Read first the whole exam, and make a strategy for which exercises you attempt first. Start with those you feel comfortable with!

Exercise 1: (1 point) Prove that $f(x,y) = (e^x + e^y, e^x + e^{-y})$ is locally invertible at every point $(x,y) \in \mathbb{R}^2$. Moreover, if $f(a) = b$, what is the derivative of f^{-1} at b?

Exercise 2: (1.5 points) Consider the ODE $x'' + 3x' + 2x = \dfrac{1}{1+e^t}$.

a) Find the general solution of the given ODE.
b) Make a sketch of the vector field corresponding to the homogeneous equation $x'' + 3x' + 2x = 0$.
c) Is there a relationship between the vector field of part b) and the general solution of part a)?

Exercise 3: (1 point) Consider $f(t) = \displaystyle\int_t^{t^2} \frac{\mathrm{d}s}{s + \sin s}$ for $t > 1$. Compute the derivative of f.

Hint: it may be useful to write f as the composition of two functions, one of which is $(x,y) \mapsto \displaystyle\int_x^y \frac{\mathrm{d}s}{s+\sin s}$.

Exercise 4: (1 point) What is the n-dimensional volume of the region

$$\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ and } x_1 + 2x_2 + \cdots + nx_n \leq n\}?$$

Exercise 5: (1 point) Let S be a closed curve in $\mathbb{R}^2$ and C the unit circle in $\mathbb{R}^2$. Suppose that S and C are diffeomorphic. What is the 2-dimensional volume of the curve S ($\mathrm{vol}_2\, S$)? Justify your answer in full detail.

Hints and remarks: we are asking for the 2-volume, and not the 1-volume, of the 1-dimensional curve S, and not of the region enclosed by it; for this exercise you may assume that "S and C are diffeomorphic" means that there is a C^r-function f, $r \geq 1$, with C^r inverse, such that $f: S \to C$ and $f^{-1}: C \to S$.

Exercise 6: (1.5 points) Let ω be the n-form in $\mathbb{R}^n$ defined by $\omega(e_1, \ldots, e_n) = 1$, where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$. Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$ given by $v_i = \sum_{1 \leq j \leq n} a_{ij} e_j$, where the a_{ij}'s are real scalars. Prove that

a) $\omega(v_1, \ldots, v_n) = \det A$, where $A = [a_{ij}]_{i,j=1,\ldots,n}$ is the $n \times n$ matrix with elements a_{ij}.
b) $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$

Exercise 7: (1 point) Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be coordinates in $\mathbb{C}^2$. Compute the integral of $\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}y_1 \wedge \mathrm{d}x_2$ over the part of the locus of the equation $z_2 = z_1^k$ where $|z_1| < 1$, oriented by $\Omega = \mathrm{sgn}\, \mathrm{d}x_1 \wedge \mathrm{d}y_1$.

Exercise 8: (1 point) Find the flux of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r^a \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where a is a number and $r = \sqrt{x^2+y^2+z^2}$, through the surface S, where S is the sphere of radius R oriented by the outward-pointing normal.

Hint: the result is a function of a and R.